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The Gini coefficient reveals more

Summary - We revisit the well-known decomposition of the Gini coefficient into between-groups, within-groups and overlap terms in the context of two groups in which the incomes in one group may be scaled and that group's population weight modified. In this more general setting than usual, we focus on the properties of the overlap term, proving inter alia that overlap unambiguously reduces as a result of a within-group progressive transfer, and is increased by scaling up the incomes in the group with the lower mean, reaching a maximum when the two means become the same. In the case of a socially heterogeneous population and equivalized incomes, the effect on the Gini overlap of changing the income unit is determined, along with that of adjusting the equivalence scale deflator in case the income unit is the equivalent adult (such adjustment simultaneously changing the weighting of income units). Relationships of the findings to existing literature are thoroughly explored.

Key Words - Gini coefficient; Inequality decomposition; Gini residual.

1. Introduction

When the Gini coefficient of income inequality is decomposed into between-groups and within-groups contributions, it is well-known that a residual term arises if the subgroup income ranges overlap. What is more, the overlap term can be very significant. In his oft-cited paper on the world income distribution, for instance, Milanovic (2002) reports for 1993 an overlap term of 6.8 in a total Gini of 57.8 points for the world as a whole (page 78). For some parts of the world the overlap term even accounts for the biggest contribution to total inequality: for Latin America and the Carribean, for instance, the overlap contributes 30.3 points out of a total Gini of 55.6; and for Western Europe, North America and Oceania, 19.4 out of 36.6 Gini points (pages 68, 69). It is strange, though, that when reaching the section in his paper in which he seeks to explain levels and trends of world inequality, Milanovic lists three questions to be answered, of which none is related directly to the overlap term.([1]) Or maybe

([1]) The three questions are: i) what lies behind the very high between-country component of inequality; ii) why is the 'pure' within-country inequality component in the Gini coefficient so small, and iii) what drove the increase of 2.7 Gini points in the between-country component which was the main factor behind the increase in the overall world inequality? (op. cit., page 78).

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this is less surprising than one might think. Indeed, for both the between-group and the within-group contributions to total inequality as measured by the Gini, Milanovic explicitly relies on the analytical expression for these terms, and is able to interpret the changes between 1988 and 1993 in terms of the change in the factors making up these expressions. But not so for the overlap term, simply because so little is known about it.([2]) Contrary to his detailed and deductive analysis of the changes in the between and within components, Milanovic relies on intuition along with some simulations to interpret overlap behaviour (see on), and remarks that "every synthetic index of inequality, and the Gini is no exception to that, is a very complex statistic" (page 80). Needless to say, then, there is plenty of room - and need - for some more analytical underpinning of the behaviour of the Gini overlap term. That is the purpose of our paper.

The Gini decomposition was first explored by Bhattacharya and Mahalanobis (1967), where an interpretation for the residual R was given in terms of concentration areas (page 150). Pyatt (1976) found an interpretation in terms of the expected value of a game, claiming an extension of existing understanding "if only at the level of psychological novelty" (page 251). Mookherjee and Shorrocks (1982) complained of R as an "awkward interaction effect ... impossible to interpret with any precision, except to say that it is the residual necessary to maintain the identity" (page 889). An interpretation in terms of reranking can be found in Silber's (1989) matrix-theoretic study (page 112). In Lambert and Aronson (1993), R is shown to measure a sub-area of the Lorenz diagram. In Giorgi (1990, 1993) can be found much detailed background material and also an interesting history of the Gini decomposition. In Zagier (1983), upper and lower bounds for the overall Gini in terms of the component Ginis are investigated. The literature of the 1970s and beyond has also seen the emergence of "decomposable" inequality indices for which no residual term arises.([3])

In analyzing the Gini decomposition here more deeply than before, we shall take the opportunity also to provide results for cases in which mutually exclusive and exhaustive subgroups are deemed "relevantly different" (to use the language of Cowell, 1980, who first set such an agenda, albeit for decom-

([2]) The closest Milanovic gets (and it is a fair attempt) is to link the overlap term with the discussion of the within-group component: "Thus any [within] inequality above 0 will 'feed' the overlap component and detract from 'within' component. Or, in other words, the overlap component will be small only if i) mean incomes are very far (different) from each other, or ii) individual country distributions are very equal." (op. cit., page 83). And consider also this: "Another question [...] is, how sensitive world Gini is to distributional changes within countries [...]. The answer is that it is sensitive although most of the change may occur through the overlap component" (page 83 again).

([3]) See Bourguignon (1979), Shorrocks (1980), Cowell (1980), Cowell and Kuga (1981) and Shorrocks (1984) for the development of these indices, Kuga (1980) for consideration of their "alikeness" with the Gini coefficient, and Ebert (1988) for a characterization of the Gini and generalized entropy family as the unique inequality indices satisfying "non-overlapping decomposability" .

posable inequality indices). We shall in fact accommodate two distinct types of transformation in this paper, in order to have flexibility in taking account of relevant differences. One is to scale the incomes of one group relative to the other (Cowell cites differences in family size or in price levels between the groups as cases where scaling may be appropriate). The other is to change the importance of a group by modifying relative weightings (for example, to induce a greater sensitivity to inequality amongst the old than the young). We shall introduce parameters to enable us to effect such transformations. The scaling and weighting parameters are in principle independent, although as Ebert (1997) has demonstrated, in the case that scaling incomes in one group corresponds to equivalizing, a concomitant and identical weighting adjustment is required if a certain rich-to-poor transfer principle is to be respected by the overall inequality measure. In the degenerate case of no scaling or weighting, our methodology generates results for the traditional decomposition of the Gini coefficient over a homogeneous population. We confine attention throughout for simplicity to the case of two population subgroups, but little of substance is lost by this.

2. Notation and preliminaries

Let there be two subgroups, which we shall call "a" and "b". These could stand for regions or any other socioeconomic partition of the population. In some of what follows, the two groups will be termed "singles" and "couples", a convenient way of referring to the special case in which "a" and "b" represent subpopulations comprising households with different needs (e.g. based on household size). As another convenience of language, we shall describe income units as "households" throughout the paper.

Let there be N_a households of type a, and N_b households of type b. Let $F_a(x)$ and $F_b(x)$ be the distribution functions, and $f_a(x)$ and $f_b(x)$ the density functions, for money income x in the two groups. We shall suppose that the distribution functions are continuous on $[0, \infty)$. Let

$$\mu_a = \int [1 - F_a(x)]dx \quad \text{and} \quad \mu_b = \int [1 - F_b(x)]dx \tag{1}$$

be the respective means, and G_a and G_b the Gini coefficients, for money income, where

$$\mu_a G_a = \int [1 - F_a(x)]F_a(x)dx \quad \text{and} \quad \mu_b G_b = \int [1 - F_b(x)]F_b(x)dx \tag{2}$$

We suppose that the money incomes of members of group "b" are deflated by a factor m relative to those in group "a" for the purposes of aggregation. In the literal case of singles and couples, the scaling could represent equivalizing.

Indeed, we shall often refer to scaled income as "living standard" in what follows, again for convenience of language in general. Thus a household of type a with money income x has living standard $y = x$ and a household of type b with money income x has living standard $y = \frac{x}{m}$.

We shall also make a weighting modification in group "b", designed to accord with the business of resort to "equivalent adults" in the living standards scenario (Ebert, 1997), as already intimated, but also to cater for other concerns, such as the one expressed by Ravallion (2004) in the context of inequality decomposition across countries: "Some sort of hybrid weighting scheme is called for, derived from an explicit assumption on the weight one attaches to country identity in assessing individual welfare ... the appropriate weights will be products of population weights and these country-specific factors" (page 13).

Thus, we create an artifical, merged population of $N_a + qN_b$ income units, one from each household of type a and q from each type b household. The parameter q, which need not be an integer, adjusts the numerical importance of the respective types in the overall population. In the literal case of singles and couples, by setting $q = 1$ we would focus on household living standards; $q = 2$ would correspond to the per capita distribution of living standards, and $q = m$ would give us the distribution of living standards among equivalent adults, fictional beings of whom there are m in each couple-household and 1 in each single-person household. For ease of language, henceforth we shall also call our artificial income units "fictional adults" in the general case.

Thus let

$$\theta = \frac{N_a}{N_a + qN_b} \tag{3}$$

be the proportion of type a households in the merged population of fictional adults. The distribution function $H(y)$ for living standards y in the merged population is defined by

$$H(y) = \theta F_a(y) + (1 - \theta)F_b(my) \tag{4}$$

The mean μ_Y and Gini coefficient G_Y for living standards among fictional adults satisfy

$$\mu_Y = \int [1 - H(y)]dy = \theta\mu_a + \frac{(1-\theta)\mu_b}{m} \quad \text{and} \quad \mu_Y G_Y = \int [1 - H(y)]H(y)dy \tag{5}$$

If we denote overall inequality when X is the money income distribution, and a deflator m and weighting factor q are used as above, as $I(X, m, q)$ in general, then in our particular case, we have $I(X, m, q) = G_Y$. Such a measure $I(\bullet, m, q)$ is partially symmetric for X in Cowell's (1980) terminology (except in the degenerate case $m = q = 1$ when it is fully symmetric). Cowell focused on decomposable inequality indices; we proceed here in terms of the Gini.

3. The Gini decomposition

The central results in this paper all come by substituting from (2) into (5), enabling us to express G_Y in terms of G_a and G_b. The following is easily verified:

$$\mu_Y(1+G_Y) = \theta^2\mu_a(1+G_a) + (1-\theta)^2\frac{\mu_b}{m}(1+G_b) + 2\theta(1-\theta)\int[1-F_b(my)F_a(y)]dy \quad (6)$$

(Proofs of this and a number of other analytical assertions, to follow, are sketched in the Appendix). This decomposition of "mean times one plus the Gini", in which the weights on the two within-group terms and the balancing item (namely, θ^2, $(1-\theta)^2$ and $2\theta(1-\theta)$) sum to unity, has not been seen in previous literature.([4]) Let φ be the share in total living standards of type a households in the merged population,

$$\varphi = \frac{N_a\mu_a}{[N_a\mu_a + qN_b\frac{\mu_b}{m}]} = \frac{\theta\mu_a}{\mu_Y} \quad \text{and} \quad (1-\varphi) = \frac{(1-\theta)\frac{\mu_b}{m}}{\mu_Y} \quad (7)$$

The well-known Gini decomposition into between group, within group and overlap terms is this:([5])

$$G_Y = \theta\varphi G_a + (1-\theta)(1-\varphi)G_b + G_{BET} + R, \quad (8)$$

where G_{BET} is the between-groups Gini, which is formally defined in (10a) - (10b) ahead, and R is, of course, the residual term. Dividing in (6) by μ_Y,

([4]) If we measure welfare as average utility, according to the imposed utility-of-income function of a social decision-maker who attributes altruism to income units (Lambert, 1985), then the inputs to social utility are people's living standards in their group and the positions of those in the group who are less fortunate than them. For the groups "a" and "b", we have $W_a = \int y[1-kF_a(x)]f_a(x)dx = \mu_a[1-\frac{1}{2}k(1+G_a)]$ (in which $y = x$), and $W_b = \int y[1-kF_b(x)]f_b(x)dx = \frac{\mu_b}{m}[1-\frac{1}{2}k(1+G_b)]$ (in which $y = \frac{x}{m}$) respectively; and overall, $W_Y = \int y[1-kH(y)]h(y)dy = \mu_Y[1-\frac{1}{2}k(1+G_Y)]$, where $h(y)$ is the density function corresponding to $H(y)$. In all of these, the parameter $k \leq 1$ measures the strength of the altruism motive relative to that of own living standard. See Lambert (2001, pp. 124-5) for more on this. Now write $W_a = \mu_a - E_a$, $W_b = \frac{\mu_b}{m} - E_b$ and $W_Y = \mu_Y - E_Y$, so that the E's measure respectively the welfare costs of inequality among type a households, among type b households and among fictional adults overall. In these terms, (6) comes down to $E_Y = \theta^2E_a + (1-\theta)^2E_b + 2\theta(1-\theta)V$, where $V = \frac{1}{2}k\int[1-F_b(my)F_a(y)]dy$. Hence (6) tells us that the cost of inequality in the merged population of fictional adults is a weighted average of within-group components and a between-groups term.

([5]) For a review of other subgroup decompositions of the Gini coefficient, which have variously been attempted using different weights, but have gained no favour, see Das and Parikh (1982, pp. 30-34).

and using (7), the implied decomposition of $1 + G_Y$ begins similarly, but has a final term which evidently subsumes the between-group and overlap effects:

$$1+G_Y=\theta\varphi(1+G_a)+(1-\theta)(1-\varphi)(1+G_b)+2\theta(1-\theta)\frac{\int[1-F_b(my)F_a(y)]dy}{\mu_Y} \quad (9)$$

The overlap term R in (8) is at once a between-groups and a within-groups effect: it measures a between-groups phenomenon, overlapping, that is generated by inequality within groups. Mishra and Parikh (1992) call R the "across-groups" contribution to the Gini coefficient, which echoes Nygård and Sandström (1981), for whom $G_{BET} + R$ is the across-groups component and who reserve the term "between groups" for entropy indices for which subgroup means tell the whole between-groups story (page 312). Mookherjee and Shorrocks (1982) complain, in respect of the residual R, that "the way in which it reacts to changes in the subgroup characteristics is so obscure that it can cause the overall Gini value to respond perversely to such changes" (page 891). Shorrocks and Wan (2004) call R a "poorly specified" element of the Gini decomposition. Milanovic (2002), in contrast, seems comfortable with the overlap contribution to the Gini decomposition, describing it in the context of world inequality analysis as measuring the degree of homogeneity within regions: "the more important the overlapping component ... the less one's income depends on where she lives" (page 70). Milanovic also attributes an increase in world overlap over time to the changing situations in India and China, occurring as "more people from these poor countries 'mingle' with people from rich countries" (page 84). By comparing the right hand sides of (8) and (9), a transparent analytical expression for R obtains, rendering this residual amenable to detailed and formal investigation, perhaps for the first time.(6)

4. Properties of the Gini overlap term

By straightforward geometry using the Lorenz diagram that obtains when each income is replaced by the mean for its group, and then using (8)-(9), it follows that:

$$\mu_a \geq \frac{\mu_b}{m} \Longrightarrow G_{BET}=\varphi-\theta \quad \text{and} \quad R=2\theta(1-\theta)\frac{\int[1-F_b(my)]F_a(y)dy}{\mu_Y} \quad (10a)$$

(6) In the case $m = 1$, the integral in (9) has close ties to Mehran's (1975) "inequality across two distributions"(pages 146-7) and to Gini's concept of transvariation, on which see Dagum (1997) and Deutsch and Silber (1997). In Yitzhaki (1988) and Yitzhaki and Lerman (1991), a covariance-based decomposition of the Gini coefficient has been developed, in which the overlap term has been analyzed deeply and shown to measure stratification among socioeconomic groups.

and

$$\mu_a \leq \frac{\mu_b}{m} \Longrightarrow G_{BET} = \theta - \varphi \quad \text{and} \quad R = 2\theta(1-\theta)\frac{\int[1-F_a(y)]F_b(my)dy}{\mu_Y} \qquad (10b)$$

(again see the Appendix). We can identify several interesting properties of the overlap from equations (8)-(9) and (10a)-(10b).([7]) These concern situations of no overlap between the subgroup income ranges, and also the effects on R of within-group money transfers, of scaling the incomes in one group relative to those in the other, and of changing the weighting parameter q in the nominated group "b". We deal with these issues in turn.

4.1. No overlap between the subgroup income ranges

As is well-known, the Gini residual R is zero if the income ranges of the two groups do not overlap. This result can readily be seen from (10a)-(10b). Set $m = 1$ first for simplicity, so that there is no scaling. If there is no overlap, clearly $\mu_a \neq \mu_b$; if $\mu_a < \mu_b$ then $F_b(y)F_a(y) = F_b(y)\ \forall y$ and if $\mu_a > \mu_b$ then $F_b(y)F_a(y) = F_a(y)\ \forall y$ (just consider the configuration of subgroup distribution functions in the two cases). These reduce to $[1 - F_b(y)]F_a(y) = 0$ and $[1 - F_a(y)]F_b(y) = 0$ respectively, and so $R = 0$ from (10a) or (10b). The same argument exactly works in the presence of scaling ($m \neq 1$) if it is the subgroups' living standard ranges which do not overlap.

4.2. Within-group money transfers

Now let one of the component income distributions change. Specifically, consider these two scenarios, in which rich-to-poor money income transfers take place within one group: (a) $F_a(x)$ changes, to $\bar{F}_a(x)$ say, which has the same mean and Lorenz dominates, with $F_b(x)$ held fixed; and (b) $F_b(x)$ changes to $\bar{F}_b(x)$, which has the same mean and Lorenz dominates, with $F_a(x)$ held fixed.

We can determine the effects of these transfers on R more easily from (8)-(9) than from (10a)-(10b). First, the effects on the overall Gini coefficient for living standards are these, from (8):

$$\Delta_a G_Y = 2\varphi.\Delta G_a + \Delta_a R \quad \text{and} \quad \Delta_b G_Y = (1-\theta)(1-\varphi).\Delta G_b + \Delta_b R \qquad (11)$$

([7]) The expressions in (10a) and (10b) coincide if $\mu_a = \frac{\mu_b}{m}$ because $\int[1 - F_a(y)]F_b(my)\,dy - \int[1 - F_b(my)]F_a(y)\,dy = \int[1 - F_a(y)]\,dy - \int[1 - F_b(my)]\,dy = \mu_a - \frac{\mu_b}{m}$.

respectively, since in each case G_{BET} is unaffected. The first term in each expression is clearly negative. First difference in (9), now, and compare with (11). Regardless of the configuration of means, the changes in R are, respectively,

$$\Delta_a R = 2\theta(1-\theta)\frac{\int F_b(my)[F_a(y) - \bar{F}_a(y)]dy}{\mu_Y};$$
$$\text{and}\quad \Delta_b R = 2\theta(1-\theta)\frac{\int [F_b(my) - \bar{F}_b(my)]F_a(y)]dy}{\mu_Y} \qquad (12)$$

in the two scenarios. Now it is well-known from dominance theory that the functions

$$S_a(y) = \int_0^y [F_a(x) - \bar{F}_a(x)]dx \quad \text{and}$$
$$S_b(y) = \int_0^y [F_b(x) - \bar{F}_b(x)]dx \ \ \forall y \qquad (13)$$

satisfy the following properties:

$$S_a(y) \geq 0, \forall y \ \&\ S_a(z) = 0 \quad \text{and}$$
$$S_b(y) \geq 0, \forall y \ \&\ S_b(z) = 0, \qquad (14)$$

where z is any income level in excess of the highest present in either subdistribution before any transfers take place (see Atkinson, 1970, or Lambert, 2001 pages 52-55, on this). Hence, re-expressing (12) in terms of $S'_a(y)$ and $S'_b(my)$ respectively, and using integration by parts, we have

$$\Delta_a R = -2m\theta(1-\theta)\frac{\int S_a(y)f_b(my)dy}{\mu_Y} \quad \text{and}$$
$$\Delta_b R = -\frac{2\theta(1-\theta)}{m}\frac{\int S_b(my)f_a(y)dy}{\mu_Y} \qquad (15)$$

both of which are non-positive. Therefore overlap cannot rise as the result of a within-group rich-to-poor money transfer.

In fact we can say slightly more. Inspecting the right-hand-side integrals in (15), we see that *overlap will fall unless the range of living standards across which* *the transfer takes place within one group is absent for the other group*.([8]) Within-group progressive money transfers thus work to reduce the Gini coefficient for living standards in the overall population of fictional adults by means of two reinforcing effects.([9]) One effect of course comes directly from the within-group inequality reduction: $\Delta G_a < 0$, respectively $\Delta G_b < 0$. To our knowledge, it has not previously been demonstrated that overlap also reduces as a result of such transfers (where these occur in the region of overlap), although the result is, of course, intuitive. Indeed, Milanovic (2002) clearly appreciates the position: "For example, if we let US, UK and German distributions experience regressive transfers ... world Gini in 1993 increases by 0.4 Gini points, 0.3 of which is due to the greater overlap" (page 83).

4.3. Scaling the incomes in one group

How does the overlap term behave if we would move one of the two subgroup income distributions "on top of" the other, by an appropriate scaling? We can examine this question most easily by setting $q = 1$ in the analytics and regarding m as simply a scaling parameter for the couples' incomes rather than specifically as an equivalence scale deflator. Let m^* be the value of m which would cause the two subgroup mean incomes to coincide:

$$m^* = \frac{\mu_b}{\mu_a}. \tag{16}$$

It seems intuitively reasonable that overlap should be at a maximum in this case. Figure 1 (taken from Lambert and Aronson, 1992) shows a situation in which this indeed happens. The graph plots the overall Gini, the within-groups contribution, the between-groups Gini and the residual, when the two sub-distributions are lognormal with a common variance of logarithms and means $\frac{\mu_b}{m}$ and μ_a, as the scaling parameter m is varied. Overlap rises sharply to a maximum at $m = m^*$, at which point the means are equated ($\frac{\mu_b}{m^*} = \mu_a$). Obviously, the between-groups Gini falls to zero at this point; the overall Gini also appears to be minimized when $m = m^*$.

([8]) When a transfer is made in a distribution F from an income unit with income v, say, to one with income $u < v$, the new distribution function $\bar{F}$ only differs from F on the range (u, v). The corresponding S-function is therefore zero outside of (u, v), and strictly positive within (u, v). Either of the integrals $\int S_a(y) f_b(my)\,dy$ and $\int S_b(my) f_a(y)\,dy$ in (15) could be zero, and would be if (and only if) the frequency density in the unaffected group were zero in the relevant range of living standards.

([9]) It is clear that G_Y must reduce, since within-group money transfers represent transfers of equivalent income in the fictional population, whatever the value of q. We shall address the issue of between-group money transfers in section 5.

To investigate the general case analytically, note that (10a) defines R when $m > m^*$ and (10b) defines R when $m < m^*$. In (10b), the numerator of R is increasing in m and the denominator is decreasing in m. It follows that $\frac{\partial R}{\partial m} > 0$ for $m < m^*$. Taking the logarithm of R and differentiating with respect to m in (10a), it follows (after a little manipulation, see the Appendix) that $\frac{\partial R}{\partial m} < 0$ when $m > m^*$. Clearly R is not differentiable at $m = m^*$. This verifies that R reaches a peak at $m = m^*$ in all cases. Letting $m \to m^*$ in either (10a) or (10b), so that $\mu_Y \to \mu_a = \frac{\mu_b}{m^*}$, the maximum value of R, call it R^*, is given as

$$R^* = 2\theta(1-\theta)\frac{\int [1 - F_b(m^* y)]F_a(y)dy}{\mu_a} \tag{17}$$

showing the dependency of the maximum value of R on the population share θ. The highest possible overlap, given the two distribution functions, occurs when $\theta = \frac{1}{2}$.

In Lambert and Aronson (1993), it was suggested that the Gini residual would generally be the higher, the closer together are the two means - and that, given the separation $S = \frac{\mu_a}{\mu_b}$ between these means, it would also be higher the larger the coefficients of variation of the two sub-distributions. The results of this section plainly accord with those speculations, proving the former of the two, and also proving that the effect of a mean-preserving spread or contraction (which of course raises or lowers the coefficient of variation) is as envisaged.

In case the two distributions differ only by scale, and are identical when $m = m^*$, as in the example upon which Figure 1 is based, we have $F_b(my) \gtrless F_a(y)$ for $m \gtrless m^*$ $\forall y$ and $G_a = G_b$. The maximum value of the overlap is then $R^* = 2\theta(1-\theta)G_a$ (from (17) and (2)). It can be verified that

$$\frac{\partial R}{\partial m} \to \theta(1-\theta)\frac{[(1-2\theta)G_a - 1]}{m^*} < 0 \text{ as } m \searrow m^* \tag{18a}$$

and

$$\frac{\partial R}{\partial m} \to \theta(1-\theta)\frac{[(1-2\theta)G_a + 1]}{m^*} > 0 \text{ as } m \nearrow m^* \tag{18b}$$

in this case (see the Appendix). For the example in Figure 1, in fact, $G_a \approx 0.276$ and $\theta = \frac{2}{3}$ (with $m^* = q = 1$).

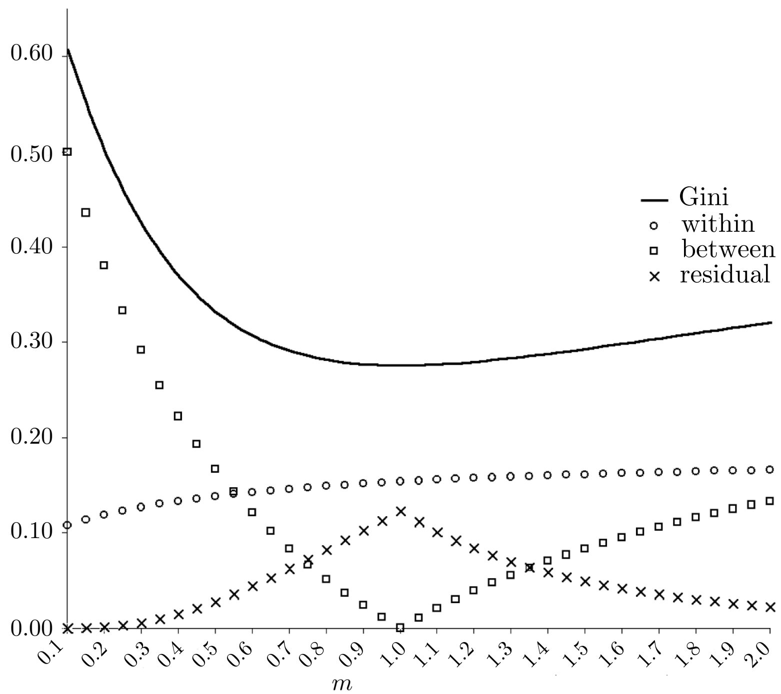

Figure 1. Overall Gini and the three components for varying m (with $m^* = 1$). Common standard deviation of logarithms of 0.5

The profiles of the within-groups term $\theta\varphi G_a + (1-\theta)(1-\varphi)G_b$, call it W, and the overall Gini G_Y suggested by Figure 1 are not specific to the particular numerical values in this example. Clearly, W and G_Y are smooth functions of m in general: $\frac{\partial W}{\partial m} = [\theta G_a - (1-\theta)G_b].\frac{\partial \varphi}{\partial m}$ and $\frac{\partial G_Y}{\partial m} = \frac{\partial W}{\partial m} + \frac{\partial(G_{BET}+R)}{\partial m}$.([10]) When the distributions differ only by scale, so that (18a) - (18b) holds, $\frac{\partial W}{\partial m} = (2\theta - 1)G_a.\frac{\partial \varphi}{\partial m}$, in which $\frac{\partial \varphi}{\partial m} = \frac{\varphi(1-\varphi)}{m}$ is positive and decreasing in m. Hence if $\theta > \frac{1}{2}$, W is increasing and concave in m (as in our example) and if $\theta < \frac{1}{2}$, $W$is decreasing and convex in m. As $m \to m^*$, $\frac{\partial W}{\partial m} \to \theta(1-\ \theta)(2\theta\ -\ 1)G_a$. Combining (18a)-(18b) with the result of differentiating in (10a) - (10b) and letting $m \searrow m^*$ and $m \nearrow m^*$ respectively, $\left.\frac{\partial(G_{BET}+R)}{\partial m}\right|_{m=m^*} = \theta(1-\theta)(1-2\theta)G_a$. Hence $\left.\frac{\partial G_Y}{\partial m}\right|_{m=m^*} = 0$,

([10]) Note by comparing (8) and (9) that $G_{BET} + R$ is a differentiable function of m in general.

as in Figure 1, whatever the values of θ and G_a. This finding could not be expected if the two distributions do not differ only by scale (see Figure 2, which is a similar plot to Figure 1, but for a case in which G_a and G_b differ).

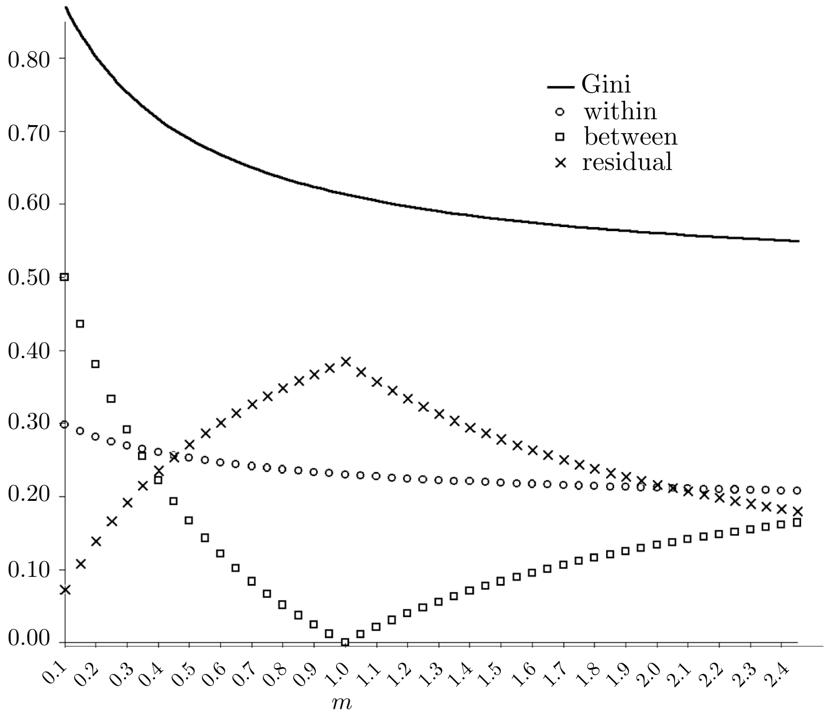

Figure 2. Overall Gini and the three components for varying m (with $m^* = 1$). Standard deviations of logarithms 0.5 and 3.0.

4.4. Changing the weighting parameter q

How will changes in the weighting parameter q affect the Gini decomposition? For example, in the case of singles and couples, suppose we move from household living standards to equivalent adult living standards, and thence to per capita living standards; each transition requires an increase in q (assuming $m < 2$, which would be usual for an equivalence scale); what happens to overlap, in particular? One's intuition may flag at this point, but the mathematics do not.

Taking logs in (10a)-(10b), differentiating with respect to q, and using $\frac{\partial\theta}{\partial q} = -\frac{\theta(1-\theta)}{q}$ and $\frac{\partial\mu_Y}{\partial q} = \frac{\mu_Y(\theta-\varphi)}{q}$ (which themselves come from differentiating in (3) and (5)), we obtain:

$$\frac{\partial R}{\partial q} = \frac{R\,(\theta + \varphi - 1)}{q} \tag{19}$$

in both cases. In other words, the elasticity of the Gini residual to changes in q is simply $\theta + \varphi - 1$. This can be positive or negative depending on the relationship between θ and φ.([11])

What is the effect on overlap of concomitantly raising the equivalence scale deflator m and weighting factor q (to maintain $q = m$)? From (10a)-(10b) it can be shown that:

$$\mu_a \geq \frac{\mu_b}{m} \Rightarrow \left.\frac{\partial R}{\partial q}\right|_{q=m} < 0 \text{ and } \mu_a \leq \frac{\mu_b}{m} \Rightarrow \left.\frac{\partial R}{\partial q}\right|_{q=m} > 0 \tag{20}$$

(details are in the Appendix). Let m^* be the equivalence scale deflator that would make the two groups equally well-off on average. If the singles are better-off than the couples ($m > m^*$), overlap in living standards of equivalent adults will fall when m is raised (along with q), and if the singles are worse-off ($m < m^*$), overlap in living standards of equivalent adults will rise when m is raised (along with q). These results are directionally the same as those for changes in m alone. Evidently, the concomitant change in the definition of the income unit q cannot overcome this effect.

5. Connections with existing literature

In this section of the paper, we reassess some existing inequality literature in light of the new information we have gained on the Gini decomposition. The papers of Paglin (1975), Foster and Shneyerov (2000), Cubel and Lambert (2002), Federov (2002) and Shorrocks and Wan (2004) all relate to scenarios in which there is no weighting of income units. They address respectively: the age-Gini, an alternative conception of within-group inequality, residual-progression-neutral income tax reforms, polarization, and the effect of changes

([11]) Two clear results are these. In the case that singles are on average better-off than their married counterparts ($\mu_a \geq \frac{\mu_b}{m}$ and $\varphi \geq \theta$), R rises when the weighting factor on couple-households is increased if singles account for more than 50% of the (weighted) population ($\theta > \frac{1}{2}$). In the case that singles are on average worse-off than couples ($\mu_a \leq \frac{\mu_b}{m}$ and $\theta \geq \varphi$), R falls when q is raised if singles are in the minority ($\theta < \frac{1}{2}$). If the two groups are equally well-off, $\frac{\partial R}{\partial q} = \frac{R(2\theta - 1)}{q}$: increasing q causes R to rise if $\theta > \frac{1}{2}$ and to fall if $\theta < \frac{1}{2}$. For the UK in 1985/6, in fact, the single were better-off than the married in equivalent income terms despite being worse-off in money income terms (Lambert, 1993) in group membership. In all cases (with $q = 1$ in our analytics), our constructions shed new light. The papers of Cowell (1980) and Ebert (1997) relate to situations in which subgroup importances are varied by means of a weighting scheme, and our analytics for these cases (in which $q \neq 1$) also turn up some new insights. Not least, in the case of group-to-group lump-sum transfers, we uncover a little-known measure of Gastwirth (1975) for earnings differentials, and its link to a more recent construction in the same vein of Milanovic and Yitzhaki (2002) which assesses relative well-being between countries or groupings of countries.

Turning first to cases for which $q = 1$, so that the importance attributed to each group depends only upon its population weight (and income share, since we are dealing here with the Gini), let the proportion of type a households be p:

$$p = \frac{N_a}{N_a + N_b} \tag{21}$$

so that $p = \theta$ when $q = 1$. We turn subsequently to scenarios in which group importances may be varied through use of the parameter q.

There are two immediate applications for the case we have just considered, of scaling the incomes in one sub-distribution relative to those in the other.

First, consider a residual-progression-neutral income tax reform, from which one group benefits and the other loses, shown in Cubel and Lambert (2002) to be welfare-improving in a wide set of circumstances despite introducing horizontal inequity. In such a case, the post-tax incomes in one sub-distribution are scaled relative to those in the other, narrowing relative income differentials between the groups on average. Let $t(x)$ be the original income tax schedule. If the groups a and b are such that $\mu_a < \mu_b$, we consider a small residual-progression-neutral tax cut in group a, and hike in group b, the post-tax incomes becoming $(1+\rho)[x-t(x)]$ in a and $(1-\lambda)[x-t(x)]$ in b where ρ and λ are such that $\theta\rho\mu_a = (1-\theta)\lambda\mu_b$ (i.e. total income tax revenue remains the same). This reform does not affect inequality within either group, and reduces inequality between groups provided $(1+\rho)\mu_a$ and $(1-\lambda)\mu_b$ are closer together than μ_a and μ_b. From (8) and (10a), the effect on the overall Gini coefficient is $\Delta G_Y = \Delta\varphi\,[\theta G_a - (1-\theta)G_b - 1] + \Delta R$ (where, since $\varphi = \theta\dfrac{\mu_a}{\mu_Y}$, we have $\Delta\varphi = \rho\varphi$). Thus the redistributive effect of the tax change, as measured by the reduction in the Gini coefficient, has two components, a positive one, $\Delta\varphi\,[1 + (1-\theta)G_b - \theta G_a]$, stemming from the narrowing of the income gaps on average, and a negative one, $-\Delta R$, stemming from the increased horizontal inequity (recalling that $\Delta R > 0$ since the reform moves the two means closer together).([12])

([12]) If any income value $x - t(x)$ is common to the two groups before imposition of the tax change, As a second application of the scaling exercise, note that the scaling when $m = m^*$ is à la Foster and Shneyerov (2000), according to whom the resultant overall Gini coefficient, call it $G_Y|_{m=m^*}$, would be the appropriate measure of within-groups inequality in the unscaled overall distribution, were the Gini to be a path-independent inequality measure. Setting $m = 1$ in (8), so that there is no scaling, we have

$$G_Y = p\varphi G_a + (1-p)(1-\varphi)G_b + G_{BET} + R. \tag{22}$$

Setting $m = m^*$ (in which case, φ becomes equal to p) in (9), we find that

$$G_Y|_{m=m^*} = p^2 G_a + (1-p)^2 G_b + R^* \tag{23}$$

where R^* is the maximum overlap. Then, subtracting,

$$G_Y - G_Y|_{m=m^*} = G_{BET} + (\varphi - p)[pG_a - (1-p)G_b] - [R^* - R] \tag{24}$$

which is not a pure between-groups term, confirming that the Gini is not path-independent. The term $G_Y - G_Y|_{m=m^*}$ subsumes the traditional between-groups Gini, but is "polluted" both by traditional within-group inequalities (unless G_b happens to equal $\frac{p}{1-p}G_a$) *and* by overlap (to the extent that this is not already maximal).

In the case that "*a*" and "*b*" represent two age groups, rather than two different household types, $G_Y - G_{BET}$ measures the so-called Paglin-Gini, call it G_{PAG}, proposed by Paglin (1975) as appropriate for capturing non-age-induced inequality in longitudinal studies. Labelling the groups such that $\mu_a \leq \mu_b$ without loss of generality, and setting $m = 1$ for no scaling, so that (10b) defines the overlap term R, we have from (8) that:

$$\begin{aligned} G_{PAG} = G_Y - G_{BET} &= p\varphi G_a + (1-p)(1-\varphi)G_b \\ &+ 2p(1-p)\frac{\int [1 - F_a(y)]F_b(y)dy}{\mu_Y}. \end{aligned} \tag{25}$$

The publication of Paglin's subsequently much-cited paper led to a string of comments, all with replies by Paglin (1977, 1979, 1989). Whilst Paglin agreed with Nelson (1977) that in (25), $p\varphi G_a + (1-p)(1-\varphi)G_b$ clearly measures inequality *within* the age groups, he disagreed with Nelson over the presence of the interaction term, eloquently defending its inclusion in his measure on

unequal treatment of equals – and reranking, a change in overlap - is inevitably introduced by the reform. See Cubel and Lambert (2002) for more on such tax reforms, and Lambert (2001, chapter 10) for more on reranking per se.

welfare grounds (Paglin, 1977, p. 523). In an exchange with Wertz (1979), the idea of measuring non-age-induced inequality as $G_Y \big|_{m=m^*}$ rather than G_{PAG} was mooted, where m^* is the scaling factor which would remove age effects on average, but not advocated (Paglin, 1979, p. 674). As is evident from (23), this alternative measure would overstate interaction effects (albeit with changed weights on the constituent terms).([13])

Let "*a*" and "*b*" be any two groups. Shorrocks and Wan (2004) point out that, so long as there is an overlap between the group income ranges, a simple distributional change can always be devised that will preserve G_Y and increase G_{BET}. Again label the groups such that $\mu_a \leq \mu_b$ and set $m = 1$. Choose two income units in the overlap, one in "*a*", having income u say, and one in "*b*" having income $v < u$. These exist or else the overlap is empty. Now simply permute the two income levels, so that the group memberships of the income units concerned are effectively swapped (equivalently, add $u - v$ to v and take $u - v$ away from u). G_{BET} is raised by this (since μ_a falls and μ_b rises), which is Shorrocks and Wan's point. Our analytics allow us to say definitively that R is reduced (as is intuitive). From (10b) the effect on R is

$$\Delta R = 2p(1-p) \int \frac{\Delta\{[1 - F_a(y)]F_b(y)\}dy}{\mu_Y} \tag{26}$$

since μ_Y is invariant to the permutation. This effect is negative since $F_a(y)$ rises and $F_b(y)$ falls for $y \in [v, u]$ and both are unaffected for $y \notin [v, u]$.([14])

Fedorov (2002) analyzes regional inequality for Russia using G_{BET}, and also studies polarization in terms of the measures presented in Wolfson (1994) and Esteban and Ray (1994) which have close connections with the Gini coefficient. The trends in interregional inequality and polarization were found to be "remarkably similar" by Fedorov (page 449), prompting him to wonder if the two polarization measures are capable of yielding additional insight.([15]) Esteban et al. (1999) propose an extension of the Esteban and Ray (1994) measure, having an additional (negative) term involving $G_Y - G_{BET}$, thereby

([13]) If incomes changed equiproportionately from the first age cohort to the second, so that $G_a = G_b$ and $R^* = 2p(1-p)Ga$ as earlier, then $G_Y|_{m=m^*} = G_a$ from (23), whilst G_{PAG} would still contain interaction effects. If the two age-groups had identical income distributions (so that $m^* = 1$ and $\varphi = p$), then G_{PAG} would perfectly capture within-age-group inequality only: $G_{PAG} = G_a$.

([14]) The effect on within-group inequality, which equals $-[\Delta G_{BET} + \Delta R]$ from (9) since $\Delta G_Y = 0$, may be positive or negative. For small income changes, the component Gini coefficients G_a and G_b go up or down depending on the ranks $F_a(u)$ and $F_b(v)$ at which the respective changes take place. Specifically, if N_a and N_b are large, then $\Delta G_a \gtrless 0$ according as $F_a(u) \gtrless \frac{1}{2}(1 + G_a)$ and $\Delta G_b \gtrless 0$ according as $F_b(v) \gtrless \frac{1}{2}(1 + G_b)$. See Lambert and Lanza (2003) for more on this.

([15]) In Duclos et al. (2004), significant differences are found between the polarization and inequality rankings of a number of countries by (a reformulation of) the Esteban and Ray (1994) measure and by the overall Gini, G_Y.

taking into account both within-group dispersion and overlap.([16]) Rodriguez and Salas (2003) show that in the case of two groups (let us say, our "a" and "b") separated by the median income, call this μ^0, the Wolfson (1994) measure equals $2\frac{\mu_Y}{\mu^0}[G_{BET} - p\varphi G_a - (1-p)(1-\varphi)G_b]$, thus capturing bi-polarization by the difference between within-group and between-group inequality (there is no overlap in this case). Zhang and Kanbur (2001) propose to measure polarization generally by the ratio of between-group inequality to within-group inequality, using a decomposable index. This effectively means that changes in polarization are determined by changes in between-group inequality and in overall inequality. If one wanted group overlap to figure explicitly in polarization, negatively of course, then the natural Gini-based version of Zhang and Kanbur's measure would be $\frac{G_{BET}}{p\varphi G_a + (1-p)(1-\varphi)G_b + R}$; if overlap should not figure at all, $\frac{G_{BET}}{p\varphi G_a + (1-p)(1-\varphi)G_b}$ could be used; variants could also be devised which would instead use $G_Y|_{m=m^*}$ in the denominator for the within-group effect.

We turn now to scenarios in which subgroup importances are varied by means of the parameter q. Ebert (1997) considered four methods of adjusting income distributions for relevant differences, corresponding in our literal scenario of singles and couples to: (i) $q = m = 1$; (ii) $q = m = 2$; (iii) $q = 2, m = m_o$; and (iv) $q = m = m_o$ (where m_o is the appropriate equivalence scale deflator, $m_o < 2$). His favoured method, (iv), the one in which equivalent adults are created,([17]) accords with a progressive transfer principle he articulates according to which a small money transfer from a household in one group to a household in the other, with a lower living standard, is welfare-improving and inequality-reducing. The effect of such a transfer on

([16]) The extended measure includes that presented in Wolfson (1994) as a special case. In both intertemporal and international comparisons, the extended measure is found by Esteban et al. to yield different conclusions than those obtaining for overall inequality using G_Y. For a discussion of how overlapping enters naturally into the measurement of polarization, see Gradin (2000, pp. 463-464). For the distinction between polarization and bi-polarization (tendency towards bimodality), see Wang and Tsui (2000), Rodriguez and Salas (2003) and Gradin (2003), in the first two of which classes of polarization indices enjoying strong connections with that in Wolfson (1994) are presented. See also Chakravarty and Majumder (2001) for a welfare-based extension of the Wolfson measure.

([17]) Fictional adults are not popular among some practitioners: "there seems little point in ... treating the family as $n^*[= m]$ units ... this appears to suggest that the importance of an individual's economic welfare is a function of the equivalence scale value of the income unit in which he or she resides ... Equivalent adults do not exist, unlike families or individuals" (O'Higgins et al, 1990, p. 26). Decoster and Ooghe (2003) discuss and compare methods (i), (iii) and (iv), using graphic examples, and go on to analyze a proposed Belgian personal income tax reform using each of the three methods. They find that method (iv) "leads to quite fanciful results with respect to the choice of equivalence scales ... although this method is normatively interesting"(page 189).

G_{BET} is clear, and the effect on the overlap R is also easy to discern for, in (10a)-(10b), both θ and μ_Y are entirely unaffected by money transfers. Suppose, then, that a small sum of money is transferred from a single (household in group a) with money income u to a couple (household in group b) with money income mv where $v < u$. For the relevant sub-intervals of $[v, u]$, we have $\Delta F_b(my) < 0$ and $\Delta F_a(y) > 0$, whilst $F_b(my)$ and $F_a(y)$ are unchanged outside of those respective sub-intervals. Hence the effect on R can be signed. If $\mu_a \geq \frac{\mu_b}{m}$, then from (10a), R goes up for such transfers (those from the on-average better-off singles to the couples), and conversely goes down for progressive transfers in the opposite direction, whilst if $\mu_a \leq \frac{\mu_b}{m}$ the effects are reversed (as is intuitive). (Compare this with the effects noted in Section 4b for within-group transfers).([18])

In Cowell (1980), who initiated the business of taking relevant differences into account in the measurement of inequality, the concept of a *uniform horizontal transfer* (UHT) is expounded, and its effect on inequality examined. This is a lump-sum transfer from each member of one group to each member of the other. For Cowell's decomposable inequality measures, the effect of a UHT is shown to be independent of intra-group distribution (page 526). Lambert (1992) shows that groupwise lump-sum transfers in an unweighted and unscaled population are overall Lorenz-improving if and only if income distribution in the donor group rank dominates that in the recipient group.

What happens in the general (weighted and scaled) case to the Gini coefficient? Suppose that an amount δ is taken from each member of group "a", and that the total, $N_a\delta$, is redistributed to the members of group "b" by lump-sum transfer, the amount received by each couple being $\frac{N_a\delta}{N_b} = \frac{p\delta}{1-p}$ (which equals $\frac{\theta q\delta}{1-\theta}$ from (3) and (21)). For assessing the effect on G_Y it is as if each of q fictional adults in the couple receives $\frac{\theta q\delta}{(1-\theta)m}$. This is not a straight transfer unless $q = m$ (one of Ebert's points).([19]) The effect on G_Y can be expressed very succinctly:

$$\frac{\mu_Y.\Delta G_Y}{\delta\theta} = (1-\theta+\theta\nu)[1-2\int F_b(my)f_a(y)dy] - (\nu-1)G_Y \qquad (27)$$

([18]) The effects on within-group inequality are hard to determine (footnote 13 is relevant here). We know, of course, from Ebert's work that overall G_Y must fall.

([19]) As an example, let $p = \frac{1}{2}$ so that the two groups have equal size, and suppose that $q = 2$ and $m = m_o = 1\frac{1}{2}$ (this case falls under scenario (iii) in Ebert (1997), which Ebert does not favour). It would be as if each marriage partner received 67 cents for every dollar donated to the couple by a single.

where $\nu = \dfrac{q}{m}$ is the ratio of the weighting factor to the scaling factor (see the Appendix). Clearly this effect depends in general on within-group inequality through the term in G_Y. For the Ebert (1997) procedure, we have $q = m = m_o > 1$ and then $\nu = 1$ in (27), so that within-group inequality is not relevant.

In the case of no weighting and no scaling, so that $q = m = \nu = 1$, (27) reduces to

$$\frac{\mu_Y.\Delta G_Y}{\delta\theta} = 1 - 2\int F_b(y)f_a(y)dy. \tag{28}$$

From this very straightforward expression, the Lambert (1992) result can be recovered and extended. The two cases

$$F_b(y) \geq F_a(y)\ \forall\, y \ \text{ and } \ F_b(y) \leq F_a(y)\ \forall\, y \tag{29}$$

are those in which group "*a*" rank dominates group "*b*" and group "*b*" rank dominates group "*a*" respectively. In these cases, from (28),

$$\begin{aligned} \frac{\mu_Y.\Delta G_Y}{\delta\theta} &\leq 1 - 2\int F_a(y)f_a(y)dy = 0 \quad \text{and} \\ \frac{\mu_Y.\Delta G_Y}{\delta\theta} &\geq 1 - 2\int F_a(y)f_a(y)dy = 0 \end{aligned} \tag{30}$$

respectively (the equalities in these being because $2\int F_a(y)f_a(y)dy = \int d[F_a(y)]^2 = 1$). We see that inequality is reduced when the donor group rank dominates, and is increased when the recipient group rank dominates.

A much weaker condition than rank dominance is, however, necessary and sufficient for inequality reduction in this special case of the Gini coefficient. Namely, $\int F_b(y)f_a(y)dy$ must be greater than $\frac{1}{2}$ - and that is all. This result may be most easily understood in the literal case of singles and couples, as follows. A single with income y would, if he or she were married to someone with no additional income, be given rank $F_b(y)$ in the distribution of couples' incomes. Taking the expectation across all singles of this artificial rank, the result may be less than, equal or greater than $\frac{1}{2}$ - which is the expected actual rank of a single among singles. If the artificial rank is greater than $\frac{1}{2}$, then singles would be placed at on-average higher ranks among couples than they actually enjoy in their own group. In this very specific sense, singles would be "on average richer" than couples - and our mathematics tells us that the group-wise transfer, being from "richer"to "poorer", would reduce inequality. This result is new, a by-product of our analysis, though the measure $\int F_b(y)f_a(y)dy$ crops up in some other literature. For example, Gastwirth (1975) uses it to quantify the earnings differential between men and women, and in Milanovic and Yitzhaki's (2002) analysis of world income inequality, $\int F_b(y)f_a(y)dy$

is used to compare relative well-being between geographical groupings of countries.([20])

6. Conclusions

The Gini coefficient is an abidingly popular and widely-used inequality index, despite many perceived problems, notably with its subgroup decomposition. When the Gini is decomposed across population subgroups, a residual term R arises, variously seen as an "awkward interaction effect" and "poorly specified" (to quote Mookherjee and Shorrocks, 1982, and Shorrocks and Wan, 2004, respectively). Through the work undertaken in this paper, we hope to have provided a path to the better understanding of the Gini decomposition, and to have thereby underscored the positive role that this index can play in certain types of inequality decomposition analysis.

We confined attention to the case of two population subgroups for ease of presentation, but the results can clearly be extended. Our model permits for the incomes in one group to be scaled, and that group's population weight modified, before measuring overall inequality and decomposing it. By this extension, we covered scenarios mooted by Cowell (1980) in which such transformations are suggested to take account of "relevant differences" between groups. We also accommodated Ebert's (1997) equivalization procedure, according to which the distribution of living standards across equivalent adults is created by concomitant scaling and weighting in one group. However, our methodology applies equally well in the absence of such scaling and weighting, when, for example, "a" and "b" could be age groups or regions.

In the general setting, we have provided simple analytics to quantify the Gini residual in terms of one or other of the integrals $\int[1-F_b(my)]F_a(y)dy$ and $\int[1 - F_a(y)]F_b(my)dy$, which clearly capture the interaction between groups explicitly. We went on to analyze theoretically the effects on the Gini decomposition of weighting and scaling in various scenarios. We thoroughly explored the relationships of our main findings to many of those in the existing literature,

([20]) Gastwirth interprets $1-\int F_b(y) f_a(y)\, dy = \int[1-F_b(y)] f_a(y)\, dy$, in the case that "$a$" comprises female workers and "b" males, as 'the probability that a randomly chosen woman earns at least as much as a randomly chosen man' (page 33). He reports a value of 0.255 for the overall US white working population in 1970 as against 0.243 in 1965, and also computes values in the range 0.165 to 0.307 for a variety of industries in 1970. Milanovic and Yitzhaki describe the entries $\int F_b(y) f_a(y)\, dy$ in their table 7 (p. 165), in which "a" and "b" are groupings of countries, as 'average rankings of members of one group in terms of the other' and note that if $\int F_b(y) f_a(y)\, dy > \frac{1}{2}$ then "a" can be seen as 'a richer group' than "b". Denoting this by $a \succ b$, their table 7 implies inter alia that {W. Europe & N. America}$\succ${Latin America & Caribbean}$\succ${E. Europe & former USSR}$\succ${Africa}$\succ${Asia}. The directions in which groupwise lump-sum transfers would reduce the overall Gini coefficient in these two contexts are evident.

drawing in, for example, the work of Cubel and Lambert (2002) on the redistributive effects of progression-neutral tax reforms, of Foster and Shneyerov (2000) on an alternative concept of within-groups inequality, of Paglin (1975) on the importance of the interaction term in the context of age groups, and of Federov (2002) and a number of other authors on the links between polarization and between-group inequality - as well as the aforementioned work of Cowell (1980) and Ebert (1997) concerning socially relevant group differences.

In respect of the "awkward" and "poorly specified" Gini residual R per se, we have furnished a number of results in this paper which surely go towards de-mystifying this term. In particular, we have shown that: (a) within-group rich-to-poor transfers cannot increase R, and will generally reduce it; (b) scaling up the incomes in the group with the lower mean raises R (to a maximum value which occurs when the group means become the same); (c) raising the population weight of one group relative to the other has an effect on R which can be straightforwardly determined; in the case of living standards and equivalent adults, (d) a small money transfer from a single to a couple who are worse-off increases R if couples have on-average lower living standards (and vice versa); and (e) if the equivalence scale deflator for couples is raised, thereby concomitantly changing the number of equivalent adults in a couple-household, overlap falls if the couples have a lower mean living standard than the singles (and *vice versa*).

Finally, we found a result quantifying the effect on the overall Gini of one of Cowell's (1980) uniform horizontal transfers (a groupwise lump-sum transfer). This result, which is new but has links with the work of Gastwirth (1975) and Milanovic and Yizhaki (2002), provides a normatively clear sense in which one group may be characterized as "richer" than another, and will surely be of interest to applied workers undertaking inequality decomposition analysis.

Appendix

By adding in (5), we have

$$\mu_Y(1+G_Y) = \int [1+H(y)][1-H(y)]dy.$$

Using (4), $\mu_Y(1+G_Y)$ can then be written as

$$\int [\theta(1+F_a(y)) + (1-\theta)(1+F_b(my))][\theta(1-F_a(y)) + (1-\theta)(1-F_b(my))]dy$$

and expanded. Substitute

$$\mu_a(1+G_a) = \int [1+F_a(y)][1-F_a(y)]dy$$

and

$$\frac{\mu_b}{m}(1+G_b) = \int [1+F_b(my)][1-F_b(my)]dy,$$

which come by adding in (1) and (2), to get:

$$\mu_Y(1+G_Y) = \theta^2\mu_a(1+G_a) + (1-\theta)^2\frac{\mu_b}{m}(1+G_b)$$
$$+\theta(1-\theta)\int[(1+F_a(y))(1-F_b(my))+(1+F_b(my))(1-F_a(y))]dy,$$

from which (6) is immediate.

From (8)-(9):

$$G_{BET} + R = -1 + \theta\varphi + (1-\theta)(1-\varphi) + 2\theta(1-\theta)\frac{\int[1-F_b(my)F_a(y)]dy}{\mu_Y}.$$

Now

$$\int[1-F_b(my)F_a(y)]dy = \int[1-F_b(my)]F_a(y)dy + \mu_a$$
$$= \int[1-F_a(y)]F_b(my)dy + \frac{\mu_b}{m}$$

from (1). Using (7), $2\theta(1-\theta)\frac{\int[1-F_b(my)F_a(y)]dy}{\mu_Y}$ can be written as

$$2\theta(1-\theta)\frac{\int[1-F_b(my)]F_a(y)dy}{\mu_Y} + 2\varphi(1-\varphi)$$

and as

$$2\theta(1-\theta)\frac{\int[1-F_a(y)]F_b(my)dy}{\mu_Y} + 2\theta(1-\varphi).$$

Therefore $G_{BET} + R$ can be written as:

$$G_{BET} + R = \varphi - \theta + 2\theta(1-\theta)\frac{\int[1-F_b(my)]F_a(y)dy}{\mu_Y} \tag{31}$$

and as:

$$G_{BET} + R = \theta - \varphi + 2\theta(1-\theta)\frac{\int[1-F_a(y)]F_b(my)dy}{\mu_Y}. \tag{32}$$

Given the values of G_{BET} stated in (10a)-(10b), (31) and (32) account for the expressions for R.

Take the logarithm of R and differentiate with respect to m in (10a)-(10b), using $\frac{\partial \mu_Y}{\partial m} = -\frac{\mu_Y(1-\varphi)}{m}$ (which itself comes by differentiating in (5) and using (7)):

$$m > m^* \Rightarrow \frac{\partial R}{\partial m} = \frac{R}{m}\left[1 - \varphi - \frac{\int my f_b(my) F_a(y) dy}{\int [1 - F_b(my)] F_a(y) dy}\right] \tag{33}$$

$$m < m^* \Rightarrow \frac{\partial R}{\partial m} = \frac{R}{m}\left[1 - \varphi + \frac{\int my f_b(my)\,[1 - F_a(y)]\,dy}{\int [1 - F_a(y)] F_b(my) dy}\right] > 0. \tag{34}$$

Now

$$\int my f_b(my) F_a(y) dy = \int [y f_a(y) + F_a(y)]\,[1 - F_b(my)] dy \tag{35}$$

using integration by parts. Substituting from (35) into (33), we find that $\frac{\partial R}{\partial m} < 0$ when $m > m^*$ as claimed in the text. Now let $m \to m^*$ in (33) and (34):

$$m \searrow m^* \Rightarrow \frac{\partial R}{\partial m} \to \frac{R^*}{m^*}\left[1 - \varphi^* - \frac{\int y f_a(y) F_a(y) dy}{\int [1 - F_a(y)] F_a(y) dy}\right] \tag{36}$$

$$m \nearrow m^* \Rightarrow \frac{\partial R}{\partial m} \to \frac{R^*}{m^*}\left[1 - \varphi^* + \frac{\int y f_a(y) [1 - F_a(y)] dy}{\int [1 - F_a(y)] F_a(y) dy}\right] \tag{37}$$

where φ^* is the value of φ when $m = m^*$ (i.e. $\varphi^* = \theta$ from (7) since $\mu_a = \frac{\mu_b}{m^*}$), and R^* is the maximum overlap, defined in (17). Substituting in (36) and (37) for φ^* and R^*, and using (2) along with $\int y F_a(y) f_a(y) dy = \frac{1}{2}\mu_a(1 + G_a)$ which follows from a result in footnote 4 with $b = 1$, (18a)-(18b) follow.

Again take logs in (10a)-(10b) and this time differentiate holding $q = m$, using

$$\left.\frac{\partial \theta}{\partial q}\right|_{q=m} = -\frac{\theta(1-\theta)}{q},$$

$$\left.\frac{\partial \mu_Y}{\partial q}\right|_{q=m} = -\frac{\mu_Y(1-\theta)}{q} \text{ and}$$

$$\left.\frac{\partial F_b(my)}{\partial q}\right|_{q=m} = y f_b(my).$$

From (10a),

$$\left.\frac{\partial R}{\partial q}\right|_{q=m} = \frac{R}{m}\left[\theta - \frac{\int my f_b(my)F_a(y)dy}{\int[1-F_b(my)]F_a(y)dy]}\right].$$

Substituting from (35) for the numerator integral, we have

$$\left.\frac{\partial R}{\partial m}\right|_{q=m} = -\frac{R}{m}\left[1-\theta + \frac{\int y f_a(y)[1-F_b(my)]dy}{\int[1-F_b(my)]F_a(y)dy}\right] < 0 \quad (38)$$

when $\mu_a \geq \frac{\mu_b}{m}$, and from (10b), straightforwardly, we have

$$\left.\frac{\partial R}{\partial m}\right|_{q=m} = -\frac{R}{m}\left[\theta + \frac{m\int y[1-F_a(y)]f_b(my)dy}{\int[1-F_a(y)]F_b(my)dy}\right] > 0 \quad (39)$$

when $\mu_a \leq \frac{\mu_b}{m}$. These results explain (20).

When a constant amount is added to each income in a distribution with mean μ and Gini coefficient G, the effect on μG is null since μG is an index of absolute inequality. Therefore in the UHT scenario, we have

$$\Delta\left[\mu_a(1+G_a)\right] = \Delta\mu_a = -\delta$$

and

$$\Delta\left[\frac{\mu_b}{m}(1+G_b)\right] = \frac{\Delta\mu_b}{m} = \frac{p\delta}{m(1-p)} = \frac{\theta\delta\nu}{1-\theta}.$$

Also, from (5),

$$\Delta\mu_Y = \theta\Delta\mu_a + \frac{(1-\theta)\Delta\mu_b}{m} = \theta\delta(\nu-1).$$

The distribution functions for money income after the UHT are $\overline{F}_a(x) = F_a(x+\delta)$ and $\overline{F}_b(x) = F_b(x - \frac{p\delta}{1-p})$. Hence

$$\Delta F_a(x) = \overline{F}_a(x) - F_a(x) = \delta f_a(x) \text{ and}$$
$$\Delta F_b(x) = \overline{F}_b(x) - F_b(x) = -\frac{p\delta f_b(x)}{(1-p)} = -\frac{2q\delta f_b(x)}{1-\theta}$$

provided δ is small. Thus from (6) it follows that

$$\Delta\left[\mu_Y(1+G_Y)\right] = -\theta^2\delta + (1-\theta)\theta\delta\nu - 2\theta(1-\theta)\int\Delta\left[F_b(my)F_a(y)\right]dy \quad (40)$$

in which

$$\Delta\left[F_b(my)F_a(y)\right] = F_b(my).\Delta F_a(y) + F_a(y).\Delta F_b(my)$$
$$= \delta\left[F_b(my)f_a(y) - \theta\frac{qF_a(y)f_b(my)}{1-\theta}\right].$$

After integrating by parts, (40) comes down to

$$\frac{\Delta\left[\mu_Y(1+G_Y)\right]}{\delta\theta} = -\theta + (1-\theta)\nu - 2(1-\theta+\theta\nu)\int F_b(my)f_a(y)dy + 2\theta\nu. \quad (41)$$

Now

$$\mu_Y.\Delta G_Y = \Delta\left[\mu_Y(1+G_Y)\right] - \Delta\mu_Y.(1+G_Y) = \Delta\left[\mu_Y(1+G_Y)\right] - \theta\delta(\nu-1)(1+G_Y).$$

Thus, from (41),

$$\frac{\mu_Y.\Delta G_Y}{\delta\theta} = -\theta + (1-\theta)\nu - 2(1-\theta+\theta\nu)\int F_b(my)f_a(y)dy + 2\theta\nu - (\nu-1)(1+G_Y)$$

from which (27) is immediate.

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